ALTERNATING CURRENT CIRCUIT WITH PULSATING CHARACTERISTICS

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ARMOUR INSTITUTE OF TECHNOLOGY
1911

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The alternating current
circuit with pulsating

THE
ALTERNATING CURRENT CIRCUIT
WITH
PULSATING CHARACTERISTICS

A THESIS

presented by

HAROLD W. NICHOLS

to the

PRESIDENT and FACULTY

of the

ARMOUR INSTITUTE OF TECHNOLOGY

for the degree of

ELECTRICAL ENGINEER

1911

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The Alternating Current Circuit
with
Pulsating Characteristics.

1. In all treatises on the subject of alternating currents it is assumed that the circuit resistance and inductance are constants. As a result very simple formulae are obtained for the current in the circuit when the applied electromotive force and circuit constants are given, and in the majority of cases the results obtained are perfectly satisfactory. Even when the current wave is known to be distorted it is usually possible to replace the actual wave of current by an "equivalent" wave on the basis of power consumption in the circuit, the power factor being then defined in terms of the phase displacement of the equivalent waves of e.m.f. and current.

There are, however, some cases in which this assumption is not justified, for example in circuits containing iron worked over a wide range of flux density, and in electrolytic cells, electric arcs, and machinery in which the magnetic reluctance pulsates during the revolution of the armature or other rotating parts. In the first case the permeability

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depends upon the current, as does also the effective conductance, representing power consumption by hysteresis, etc.; in the second the circuit resistance is not a constant, but varies periodically during each cycle. In each case the inductance or resistance may be considered, to a first approximation, as made up of a constant part and of a part which varies periodically with double the frequency of the fundamental of current. The amplitude of the pulsation will depend upon the physical qualities of the medium supporting the current, and also upon the upper and lower limits of the current. It is to be determined by experiment. The general problem to be solved is thus:

Given an alternating current circuit in which the circuit characteristics are representable as double-frequency pulsations of resistance and reactance, and upon which there is impressed a periodic e.m.f. of arbitrary wave-form: it is required to find the instantaneous value of current after the steady state has been reached.

It will be assumed that the impressed e.m.f. wave is a symmetrical one; that is, that there are

present no harmonics of even order. It is also assumed, as always in the study of the power applications of alternating currents, that the frequency is not too high and that the conductors considered are comparatively good ones, i.e., metallic, so that the potential energy of the electric field is to be neglected in comparison with magnetic energy due to the currents in the conductors.

2. In the given circuit let the inductance be:

$$L = L_o\{ 1 + 2h \cos 2pt \}$$

the resistance:

$$R = R_o\{ 1 + 2k \cos (2pt - \epsilon)\}$$

$$= R_o\{ 1 + 2k\epsilon'' \cos 2pt + 2k\epsilon' \sin 2pt \}$$

where $\epsilon' = \sin\epsilon$, $\epsilon'' = \cos\epsilon$, and $p/2\pi$ is the frequency of the fundamental of e.m.f. or current.

The impressed electromotive force:

$$e = \sum \{ E'_n \sin npt + E''_n \cos npt \}$$

where n has all odd integral values by the first assumption. Due to this e.m.f. there will flow in the circuit a periodic current determined by the equation:

$$\frac{d(Li)}{dt} + \frac{R(Li)}{L} = e \qquad (I)$$

and the solution of the problem consists in the solution of this equation, giving i as a function of the time.

3. The equation is linear in (Li) and the solution for the steady state is

$$\varphi = Li = \varepsilon^{-\int\frac{R\,dt}{L}}\int \varepsilon^{\int\frac{R\,dt}{L}}\, e\,dt$$

The actual performance of this integration is not practicable; an approximate solution will therefore be sought in which the degree of approximation is that warranted by the assumed forms of R and L.

Expansion of R/L gives

$$\frac{R_0}{L_0}\left\{1 + 2k\Delta - C\cos(2\omega t + \varphi) + \text{[illegible]}\right\}$$

where

$$\Delta = h - k\epsilon', \quad \varphi = \tan^{-1}\text{[illegible]}$$

$$C^2 = 4(h^2 + k^2 - 2kh\epsilon)$$

Hence:

$$\int\frac{R\,dt}{L} = \frac{R_0}{L_0}\left\{t(1+2k\Delta) + \ldots\right\} \text{[illegible]}$$

Here the terms containing trigonometric functions

involve powers of p in the denominators, and may therefore be neglected in comparison with the others. We thus put:

$$\tau = 1 + 2FA$$

which is therefore a constant not differing greatly from unity. $\frac{L_0}{R_0\tau}$ is the generalized time-constant of the circuit in the steady state, and differs from the time-constant as usually defined by the correction factor τ.

The solution of the equation is now:

$$L_0 i = \varepsilon^{-\frac{R_0\tau}{L_0}} \int \varepsilon^{\frac{R_0\tau}{L_0}} e \, dt$$

which gives

$$L_0 i = \sum \frac{L_0 E'_n}{Z_n^2}\left(R_0\tau \sin npt - npL_0 \cos npt\right) + \sum \frac{L_0 E''_n}{Z_n^2}\left(R_0\tau \cos npt + npL_0 \sin npt\right)$$

$$= L_0 \sum \frac{1}{Z_n^2}\left(F_n \sin npt + G_n \cos npt\right)$$

in which:

$$Z_n^2 = R_0^2\tau^2 + n^2p^2L_0^2$$

$$F_n = R_0\tau E'_n + npL_0 E''_n$$

$$G_n = R_0\tau E''_n - npL_0 E'_n$$

Hence it follows for the current:

$$i = \frac{1}{1 + 2h\cos 2pt} \sum \frac{1}{Z_n^2}\left[F_n, G_n\right]$$

4. In order to reduce this to periodic form the outside factor may be expanded, giving

$$1 - 2h\cos 2pt + \ldots$$

and the summation must be multiplied by this series. Now in appendix (I) it is shown that the amplitude of any harmonic of current of order m involves e.m.f. harmonics of orders m, m ± 2 and no others; hence the above expansion need not be carried beyond the second term. The current is therefore, to a first approximation:

$$i = (1 - 2h\cos 2pt) \sum_{1} \frac{1}{Z_n^2}\left[F_n, G_n\right]$$

Now, by means of the relations:

$$2 \cos m \sin n = \sin(m+n) - \sin(m-n)$$

$$2 \cos m \cos n = \cos(m-n) + \cos(m+n) .$$

the expression for i is transformed into the following:

$$i = \sum_{1}^{7} \frac{1}{\tilde{Z}_n^{2}} \left\{ F_n \sin npt + G_n \cos npt \right\}$$

$$- k \sum^{7} \frac{F_n}{\tilde{Z}_n^{2}} \left\{ \sin(n-2)pt + \sin(n+2)pt \right\}$$

$$- k \sum^{7} \frac{G_n}{\tilde{Z}_n^{2}} \left\{ \cos(n-2)pt - \cos(n+2)pt \right\}$$

Collecting the coefficients of the sines and cosines of like orders there is obtained finally:

$$i = \sin pt \left\{ \frac{F_1}{\tilde{Z}_1^{2}} + \frac{kF_1}{\tilde{Z}_1^{2}} - \frac{kF_3}{\tilde{Z}_3^{2}} \right\}$$

$$+ \sin 3pt \left\{ \frac{F_3}{\tilde{Z}_3^{2}} - \frac{kF_1}{\tilde{Z}_1^{2}} - \frac{kF_5}{\tilde{Z}_5^{2}} \right\}$$

$$+ \sin 5pt \left\{ \frac{F_5}{\tilde{Z}_5^{2}} - \frac{kF_3}{\tilde{Z}_3^{2}} - \frac{kF_7}{\tilde{Z}_7^{2}} \right\}$$

$$\text{(A)} \qquad + \ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \cos pt \left\{ \frac{G_1}{\tilde{Z}_1^{2}} - \frac{kG_1}{\tilde{Z}_1^{2}} - \frac{kG_3}{\tilde{Z}_3^{2}} \right\}$$

$$+ \cos 3pt \left\{ \frac{G_3}{\tilde{Z}_3^{2}} - \frac{kG_1}{\tilde{Z}_1^{2}} - \frac{kG_5}{\tilde{Z}_5^{2}} \right\}$$

$$+ \cos 5pt \left\{ \frac{G_5}{\tilde{Z}_5^{2}} - \frac{kG_3}{\tilde{Z}_3^{2}} - \frac{kG_7}{\tilde{Z}_7^{2}} \right\}$$

$$+ \ \ldots\ldots\ldots\ldots\ldots\ldots$$

This last equation constitutes the solution of the problem. It is seen that the current is derived from that calculated on the basis of constant inductance and resistance by the addition of corrective terms to each harmonic amplitude. If h = 0 = k these correction terms become zero and the problem reduces to the ordinary one, as it evidently should do.

5. Some general conclusions may be drawn:

a) A symmetrical e.m.f. wave gives rise to a symmetrical wave of current due to double-frequency pulsation of resistance and inductance.

b) The effect of this pulsation is to make given harmonics of current depend upon higher and lower harmonics of e.m.f. The harmonics are "mixed".

c) The wave of magnetic flux does not display this mixing of harmonics.

As a partial check upon the work just given, let h = 0 = k ; this should give the solution of the problem for the case of constant resistance Ro and constant inductance Lo. For simplicity assume but one pure sine component to the e.m.f.; it is evident that if the equations check for this they will also

for any arbitrary wave, for the solution is formed by a simple addition of terms. Thus let

$$e = E \sin pt$$

We get:

$$\tau = 1$$

$$Z^2 = R^2 + p^2 L^2$$

$$F = R E , \quad G = - p L E$$

Therefore

$$i = (R E/Z^2) \sin pt -- (p L E/Z^2) \cos pt$$

and squaring and adding the components, we get for the value of current (numerical value)

$$i = E/Z$$

which would also have been obtained directly from the simple fundamental equation, as is well known.

It is remarkable that the factor τ, although it contains terms from both the inductance and resistance coefficients, enters the equations only as a multiplier of Ro. This quantity, τRo, will be called the "apparent" resistance of the circuit (R'), and is not the same as the impedance Z. It appears from this that the effect of pulsating resistance and reactance may be in a measure taken into account by attributing to the circuit the new resistance R' instead of its actual value R. The advantage of this is that R' is a constant. This method will give

the correct value of the circuit impedance, but will not account for the mixing of harmonics.

When h = 0 = k the apparent resistance reduces to the fixed resistance, Rc, since then $\tau = 1$, so that the name is justified.

Special Cases.

6. (a) Resistance small, for example, the exciting current for a coil containing a medium of variable permeability, but for which the hysteresis losses may be neglected. This will give a distorted current wave and 90° phase displacement of the current from the e.m.f.

In this case the current is:

$$i = \frac{1}{L}\int \sum \left(\mathcal{E}_n' \sin npt + \mathcal{E}_n'' \cos npt\right) dt$$

$$= \frac{1}{pL}\sum \Big\{ \frac{1}{n}\left(-\mathcal{E}_n' \cos npt + \mathcal{E}_n'' \sin npt\right)$$

$$+ \frac{h}{n}\left[\mathcal{E}_n' \cos(n-2)pt + \mathcal{E}_n' \cos(n+2)pt\right]$$

$$- \frac{h}{n}\left[\mathcal{E}_n'' \sin(n+2)pt + \mathcal{E}_n'' \sin(n-2)pt\right]$$

and reduction to linear form gives:

$$
\begin{aligned}
i = {} & \cos pt \left\{ (h-1)\,\mathcal{E}_1' + \frac{h}{3}\,\mathcal{E}_3' \right\} \\
& + \cos 3pt \left\{ -\frac{1}{3}\,\mathcal{E}_3' + \frac{h}{5}\,\mathcal{E}_5' - h\,\mathcal{E}_1' \right\} \\
& + \cos 5pt \left\{ -\frac{1}{5}\,\mathcal{E}_5' + \frac{h}{7}\,\mathcal{E}_7' - \frac{h}{3}\,\mathcal{E}_3' \right\} \\
& + \ldots\ldots\ldots\ldots \\
& + \sin pt \left\{ (h+1)\,\mathcal{E}_1'' - \frac{h}{3}\,\mathcal{E}_3'' \right\} \\
& + \sin 3pt \left\{ \frac{1}{3}\,\mathcal{E}_3'' - \frac{h}{5}\,\mathcal{E}_5'' - h\,\mathcal{E}_1'' \right\} \\
& + \sin 5pt \left\{ \frac{1}{5}\,\mathcal{E}_5' - \frac{h}{7}\,\mathcal{E}_7'' - \frac{h}{3}\,\mathcal{E}_3'' \right\} \\
& + \ldots\ldots\ldots\ldots
\end{aligned}
$$

As a first illustration, let a pure sine wave be impressed upon the coil, and assume $h = .15$, a value which serves for transformer iron worked at a fair density. Let also:

$$E_1'' = 100\,, \qquad p\,L_o = 5.0$$

This gives for the current

$$i = 23 \sin pt - 3 \sin 3pt\,, \qquad I = 16.4 \text{ amps.}$$

The results are plotted in figure 1. Since the e.m.f. wave is a pure sine one, that of flux is also, so that the magnetization curve for the specimen may be plotted, giving the lower curve in the figure. It is seen that this method gives a natural shape for

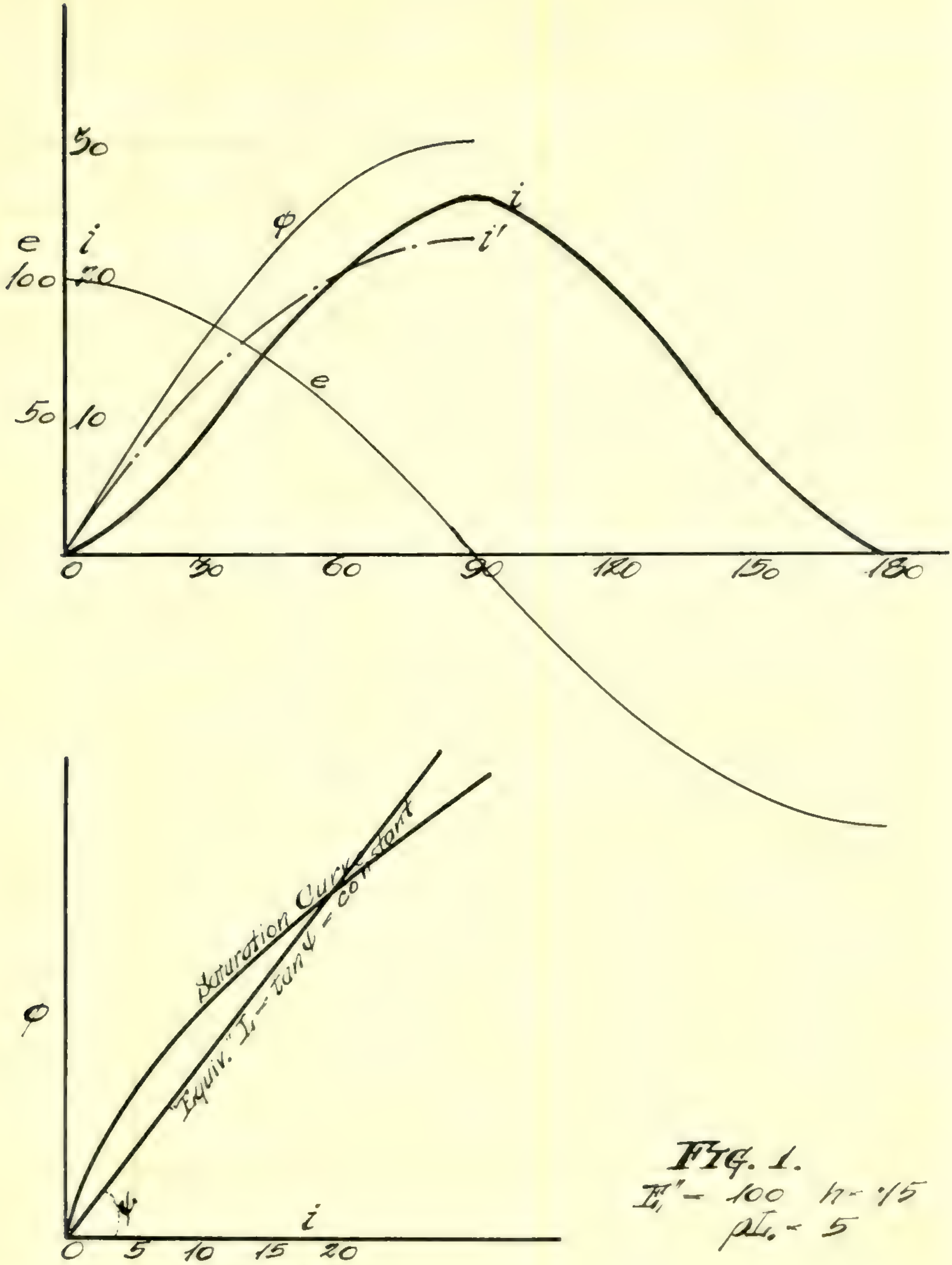


FIG. 1.

$E_1'' = 100 \quad h = .15$

$pL_0 = 5$

this curve, showing that the assumed form of L is legitimate and is able to give a closer approximation than the ordinary assumption of constant inductance and the equivalent pure sine wave. The equivalent curves for current and inductance are plotted for comparison, the maximum value of the flux being assumed arbitrarily, as depending upon the particular magnetic circuit chosen. The equivalent value of the inductance, that is, the value which would be used on the assumption that L is constant, is represented by the slope of the magnetization curve L', which is constant. The true inductance is represented by the slope of the curve L and decreases as the current increases, as it should do. Due to the variation of reactance there is introduced a third harmonic of amplitude 13% of the fundamental.

As a second example, let the impressed e.m.f. wave be a peaked one:

$$e = 100 \cos pt + 15 \cos 3pt$$

$$E''_1 = 100, \qquad E''_3 = 15, \qquad h = .15, \qquad pL_0 = 5$$

The current is

$$i = 22.85 \sin pt - 2 \sin 3pt - .15 \sin 5pt$$

$$E = 71.5, \qquad I = 16.2 \quad \text{(effective)}$$

The results are shown in fig. 2. The primed

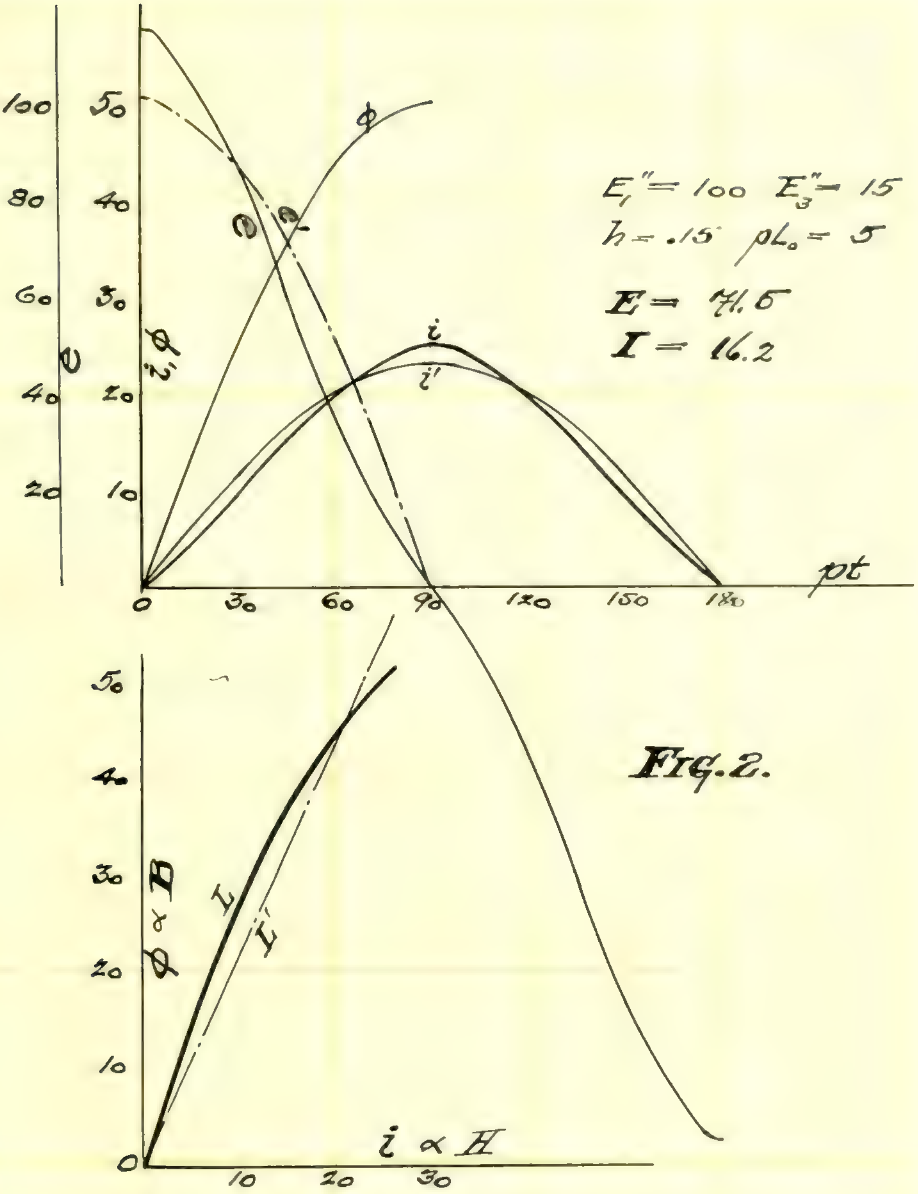
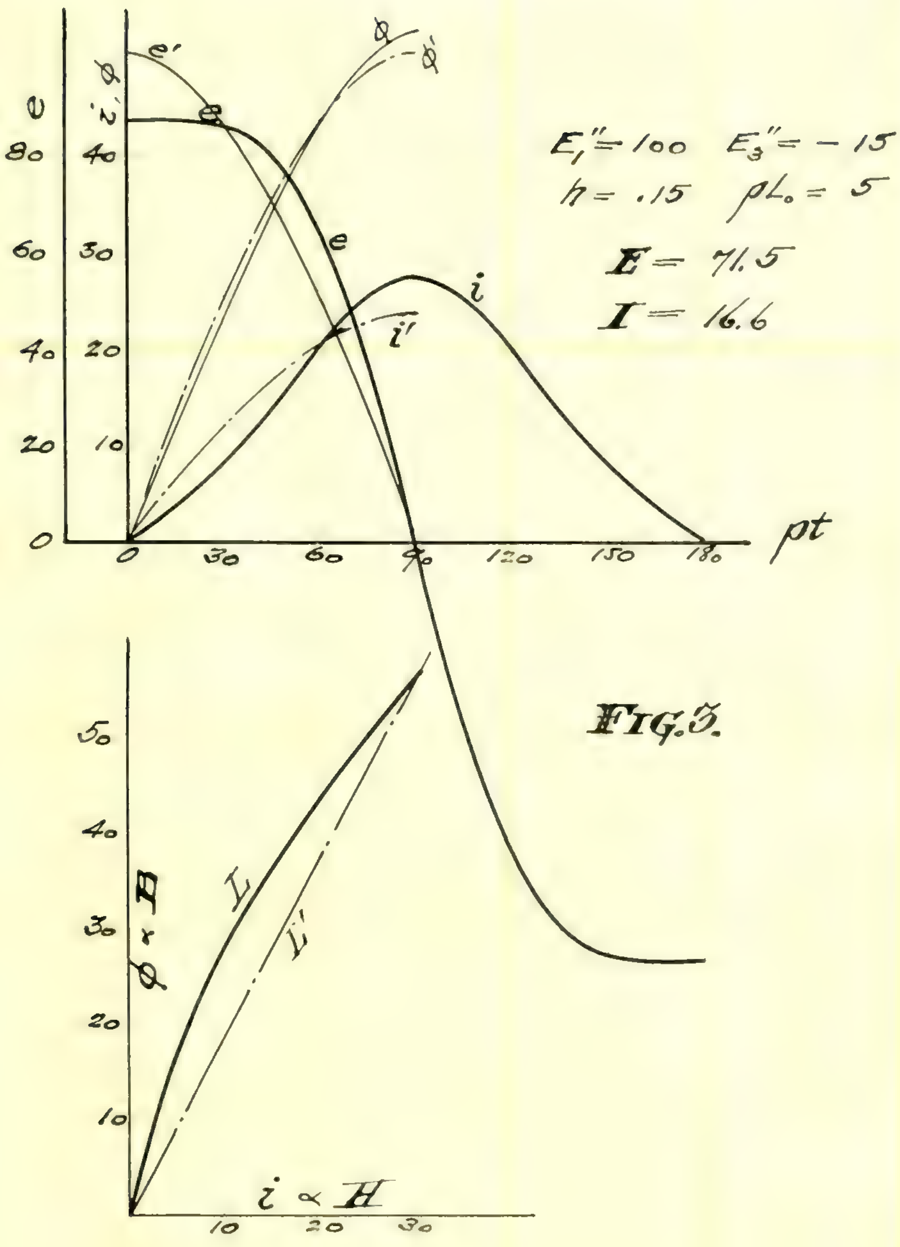

100
80
60
40
20
e
50
40
30
20
10
i, i', φ
φ
e
e'
i
i'
E₁'' = 100 E₃'' = 15
h = .15 pL₀ = 5
E = 71.5
I = 16.2
0
30
60
90
120
150
180
pt
50
40
30
20
10
0
φ ∝ B
L
L'
i ∝ H
10
20
30

FIG. 2.



e'
φ
φ'
i'
e
e
i
i'
$E_1'' = 100$ $E_3'' = -15$
$h = .15$ $pL_0 = 5$
E = 71.5
I = 16.6
e
80 60 40 20 0
40 30 20 10
0 30 60 90 120 150 180
pt
φ ∝ B
L
L'
50 40 30 20 10
i ∝ H
10 20 30

FIG. 3.

letters indicate the equivalent values.

Third, let the impressed wave be a flat one:

$$e = 100 \cos pt -- 15 \cos 3pt$$

Therefore

$$i = 23.15 \sin pt -- 4 \sin 3pt + .15 \sin 5pt$$

$$E = 71.5. \qquad I = 16.6 \qquad \text{(effective)}$$

The results are plotted in the third figure; These three cases are typical of all that might arise; they show that the form of the characteristic for the iron is independent of the particular wave form impressed (which had to be shown, since the inductance was assumed an explicit function of the time) and that the form adopted for the inductance is able to give a closer approximation than is ordinarily obtained, and without too great a sacrifice of convenience.

The waves are all symmetrical with respect to their maximum ordinates, since there are no variable energy losses.

b) Circuit of small inductance and variable resistance. The problem reduces to the solution of the equation:

$$R_0\{1 + 2k\cos 2pt\}\, i = \sum (E_n' \sin npt + E_n'' \cos npt)$$

which gives on expansion:

$$i = \frac{1+2K^2}{R_0}\sum \left(\mathcal{E}_n' \sin npt + \mathcal{E}_n'' \cos npt\right)$$

$$-\frac{K}{R_0}\sum \left\{\mathcal{E}_n'\left[\sin (n+2)pt + \sin (n-2)pt\right] + \mathcal{E}_n''\left[\quad\right]\right\}$$

$$+\frac{K^2}{R_0}\sum \left\{\mathcal{E}_n'\left[\sin(n+4)pt + \sin(n-4)pt\right] + \mathcal{E}_n''\left[\quad\right]\right\} + \cdots$$

that is:

$$R_0 i = \sin pt\left[\mathcal{E}_1'(1+2K^2) + K\mathcal{E}_1' - K\mathcal{E}_3' + \cdots\right]$$

$$+ \sin 3pt\left[\mathcal{E}_3'(1+2K^2) - K\mathcal{E}_1' - K\mathcal{E}_5' \cdots\right]$$

$$+$$

$$+ \cos pt\left[\mathcal{E}_1''(1+2K^2) - K\mathcal{E}_1'' - K\mathcal{E}_3'' - \cdots\right]$$

$$+ \cos 3pt\left[\mathcal{E}_3''(1+2K^2) - K\mathcal{E}_1'' - K\mathcal{E}_5'' + \cdots\right]$$

$$+$$

The results show that pulsation of the circuit resistance alone will cause a distortion of the current wave by the introduction of harmonics of other orders. The ones which are principally effective in distorting the wave, however, are those which are adjacent to the harmonics of e.m.f., since the others are multiplied by higher powers of k. In a circuit of small inductance, therefore, the double frequency pulsation of resistance cannot introduce a direct current component into the current; hence if this occurs it is due either to counter electromotive forces or to pulsations of odd frequency. The term taking account of counter electromotive forces has not been added in the equation, but the form will be very simple: there will be only a constant subtracted from the right hand side, which is the same thing as allowing the index n to have the value zero in addition to its odd values. An example of this distortion may be found in the case of the ordinary incandescent lamp at comparatively low frequencies.

There is no phase displacement of the current from the e.m.f., for the coefficients of the sine terms in the e.m.f. wave appear only in the sine terms of the current wave, and vice versa.

As an illustration of this distortion, let

$$E' = 100, \qquad k = .15, \qquad R_0 = 10$$

that is, a pure sine wave of impressed e.m.f. The resulting current is:

$$i = 12.05 \sin pt -- 1.5 \sin 3pt$$

and is a peaked wave of effective intensity 8.6 as against 7.1 on the assumption of constant resistance equal to the mean value, R_0. In this case it is to be noted that the standard of phase is taken to be that of the pulsating resistance.

c) R/L = constant, = w, that is,

$$k = h, \quad \epsilon' = 0$$

The exact solution is

$$(Li) = \sum \frac{1}{w^2 + n^2p^2}\Big[(w\xi_n' - np\xi_n'') \sin npt + (w\xi_n'' - np\xi_n') \cos npt\Big]$$

or, expanding the factor L:

$$i = \frac{1}{L_0}(1 - 2k \cos 2pt + \ldots) \sum \frac{J_n \sin npt + K_n \cos npt}{p^2}$$

in which:

$$J_n = w\,E'_n + n\,p\,E''_n$$

$$K_n = w\,E''_n - n\,p\,E'_n$$

$$v_n^2 = w^2 + n^2p^2$$

Reducing in the usual way there is obtained:

$$\begin{aligned} i = {} & \sin pt\left(\frac{J_1}{v_1^2} + \frac{k J_1}{v_1^2} - \frac{k J_3}{v_3^2} - \dots\right) \\ & + \sin 3pt\left(\frac{J_3}{v_3^2} - \frac{k J_1}{v_1^2} - \frac{k J_5}{v_5^2} - \dots\right) \\ & + \dots \\ & + \cos pt\left(\frac{K_1}{v_1^2} - \frac{k K_1}{v_1^2} - \frac{k K_3}{v_3^2} - \dots\right) \\ & + \cos 3pt\left(\frac{K_3}{v_3^2} - \frac{k K_1}{v_1^2} - \frac{k K_5}{v_5^2}\right) \\ & + \dots \end{aligned}$$

The results are much the same as those obtained for the case of the pulsating reactance only. In each case it is the varying inductance which produces the marked distortion of current.

7. General Case. (page 7)

The important feature of this method is that it is capable of representing, by a proper choice of the constants h, k, σ, the distorted current wave produced by hysteresis, and hence to represent analytically the hysteresis loop for magnetic materials. As an illustration some examples have been worked out.

Fig. 4 gives the current produced by a sine wave of impressed e.m.f. when h = .15 and k = .10 in a circuit whose inductance L_0 is .01 and effective constant resistance 5. The current wave is not that due to distortion by hysteresis, but to variable impressed energy supply in the medium in addition to that supplied by the e.m.f., that is to say, the "hysteresis" loop is traversed in the negative direction, as may be seen by plotting, or from the shape of the current curve.

The next figure (5) shows the state of affairs when the sign of h has been changed. The current wave has a more natural appearance, and gives a more nearly correct hysteresis loop, as shown in the lower figure. There is still a negative area, however, which is eliminated by a better choice of h as in figure 6. This value is one which comes from an inspection of the permeability curve of a specimen

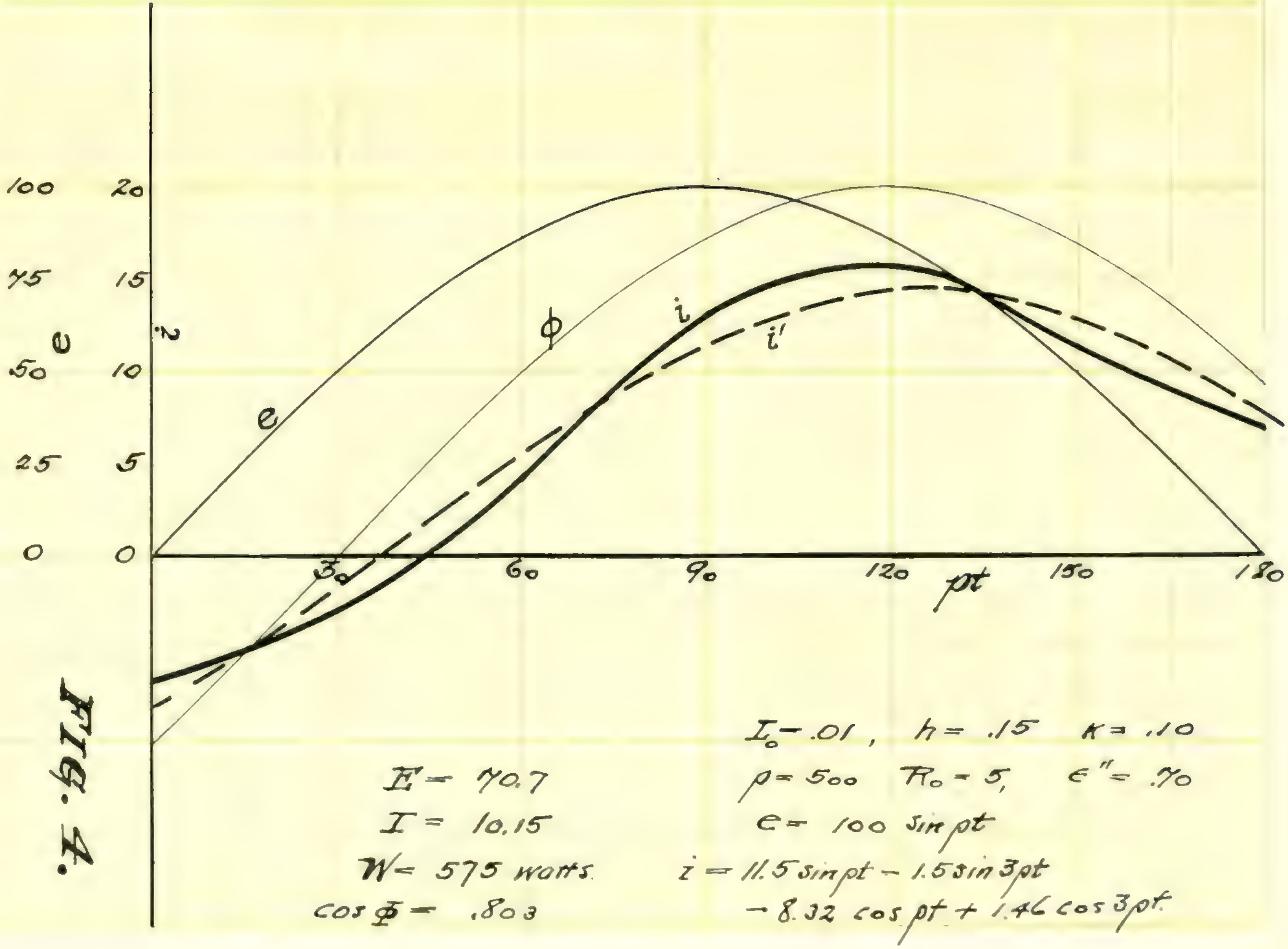

100
75
50
25
0
20
15
10
5
0
e
2
e
φ
i
i'
30
60
90
120
pt
150
180
$I_o = .01$, $h = .15$ $\kappa = .10$
$\rho = 500$ $R_o = 5$, $\epsilon'' = .70$
$e = 100 \sin pt$
$i = 11.5 \sin pt - 1.5 \sin 3pt$
$- 8.32 \cos pt + 1.46 \cos 3pt$
$E = 70.7$
$I = 10.15$
$W = 575$ watts.
$\cos \Phi = .803$

FIG. 4.

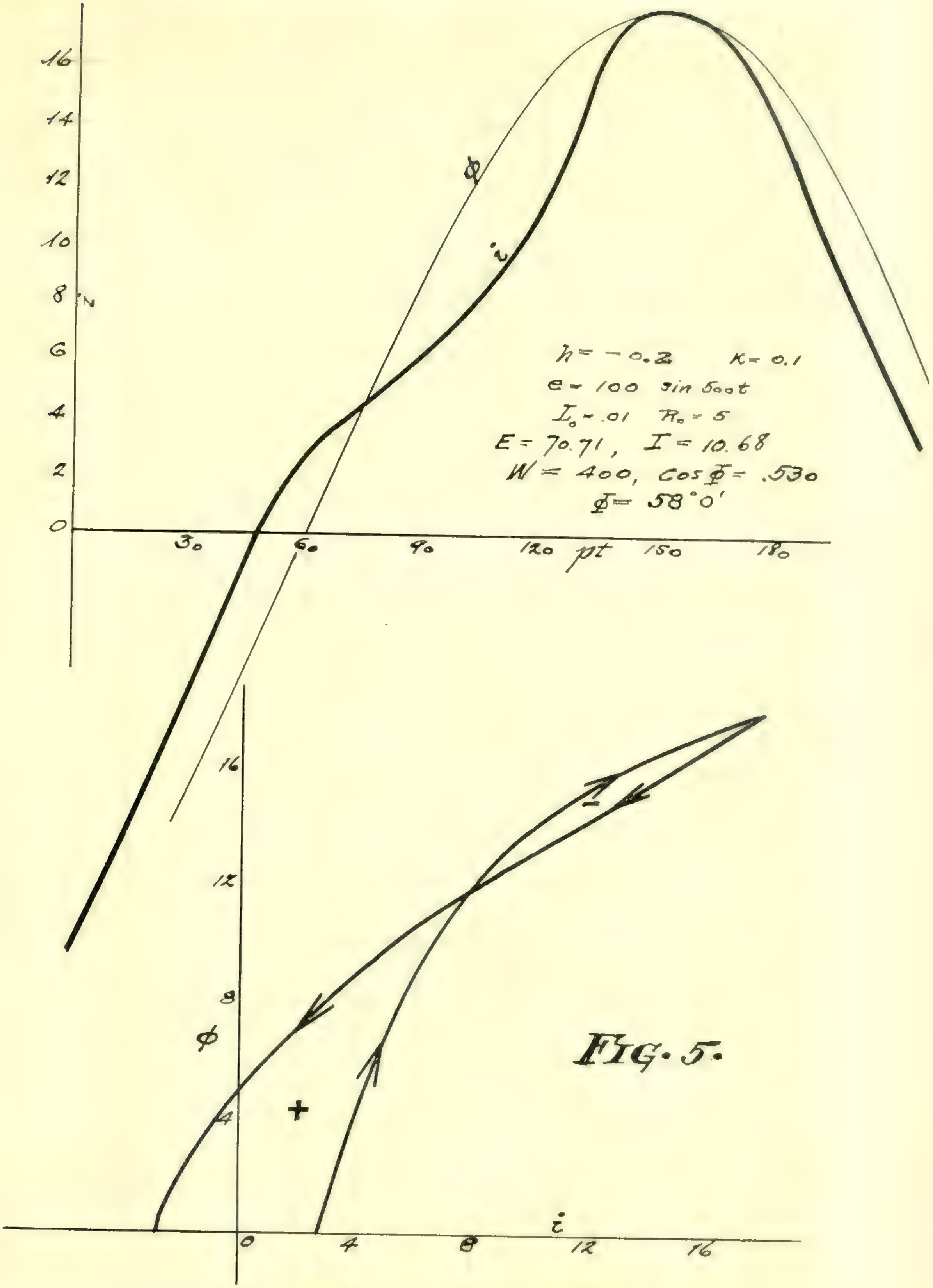

16
14
12
10
8
6
4
2
0
30
60
90
120
pt
150
180
Φ
i
h = −0.2   k = 0.1
e = 100 sin 500t
I₀ = .01   R₀ = 5
E = 70.71, I = 10.68
W = 400, cos Φ = .530
Φ = 58°0′
16
12
8
Φ
4
+
0
4
8
12
16
i

FIG. 5.

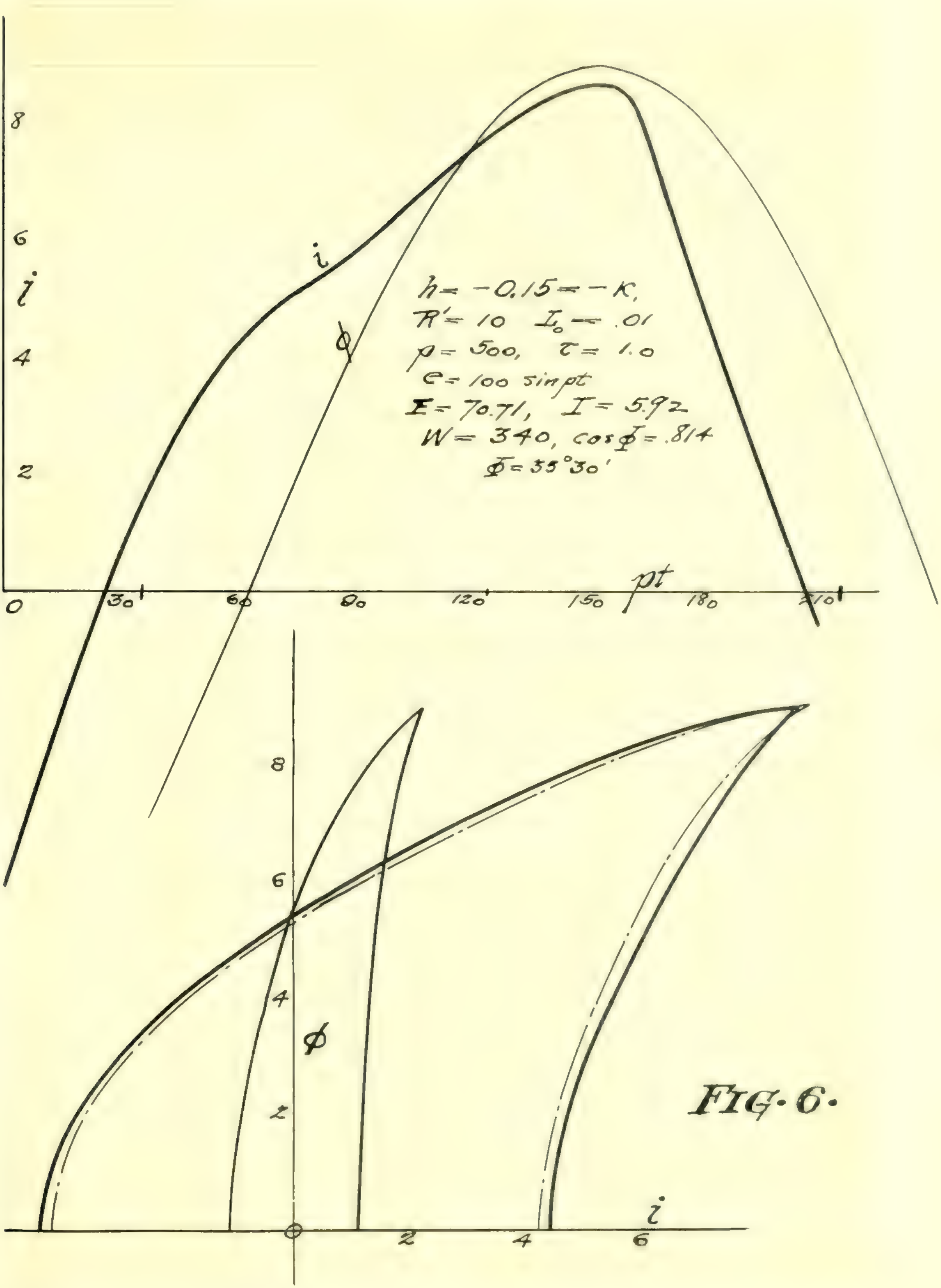
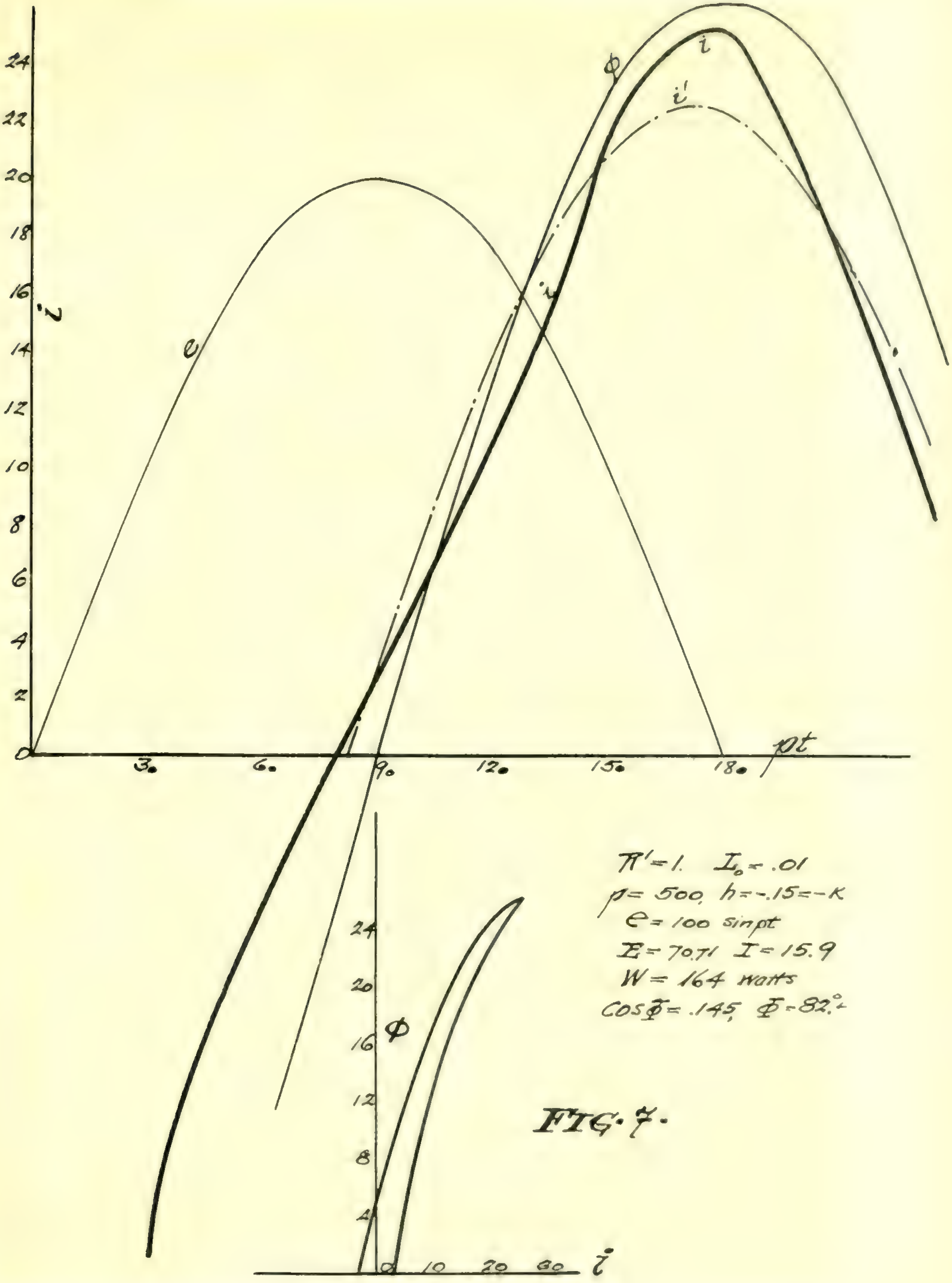

8
6
i
4
2
i
ϕ
h = −0.15 = −K,
R′ = 10 I₀ = .01
p = 500, τ = 1.0
e = 100 sin pt
E = 70.71, I = 5.92
W = 340, cos Φ = .814
Φ = 35°30′
0
30
60
90
120
150
180
210
pt
8
6
4
ϕ
2
0
2
4
6
i

FIG. 6.

of iron. The current wave now has the appearance characteristic of an iron-clad circuit, and gives a typical hysteresis loop. The dotted curve in the lower figure is the one for which $h = --.18$. In these curves the equivalent wave has been obtained by finding the sine wave which gives the same power, both dissipated and stored. This condition determines the amplitude and phase of the equivalent current, and these values are indicated or plotted in the figures.

Figure 7 shows the current and hysteresis loop when the energy loss in the circuit is rather small compared with the wattless power. The power factor is .145 and the equivalent current lags by 82°. The form of the loop is still correct.

Figure 8 shows the same thing when the power ~~f~~ factor is high-- .950 --and the lag of equivalent current 18°30'. By giving h the proper value, and also taking account of the magnitude and phase of the effective resistance, practically any loop may be reproduced. In the figures a sine wave of e.m.f. has been assumed for simplicity, but the choice of any arbitrary wave makes necessary only the addition of more terms to the current equation.



$R' = 1. \quad I_0 = .01$

$p = 500, \quad h = -.15 = -K$

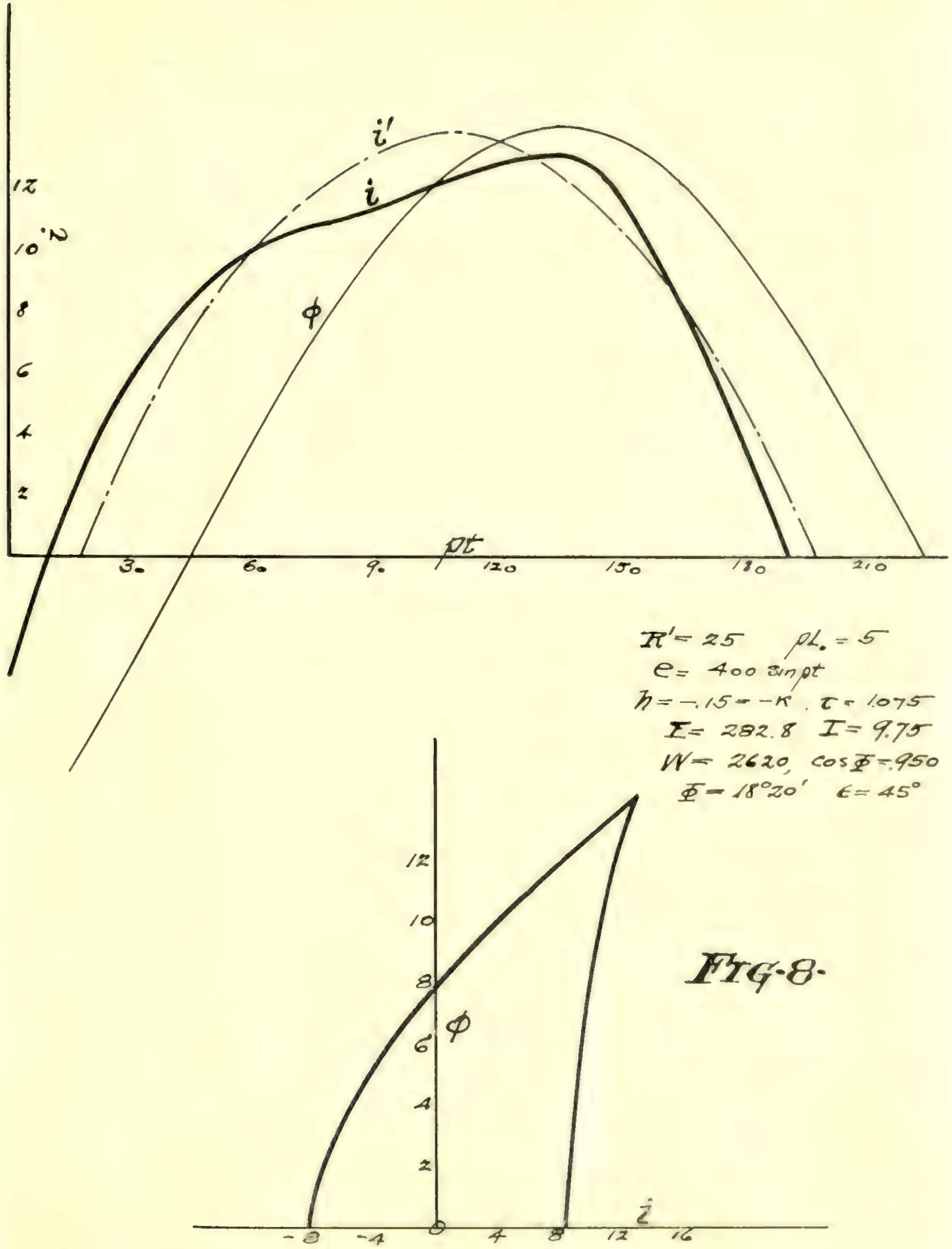
$e = 100 \sin pt$

$E = 70.71 \quad I = 15.9$

$W = 164$ watts

$\cos \Phi = .145, \quad \Phi = 82°+$

FIG. 7.



$R' = 25 \quad pL_0 = 5$

$e = 400 \sin pt$

$h = -.15 = -K, \ \tau = 1.075$

$\mathcal{E} = 282.8 \quad I = 9.75$

$W = 2620, \ \cos\Phi = .950$

$\Phi = 18°20' \quad \epsilon = 45°$

FIG-8.

It is thus possible to represent analytically, with a fair degree of approximation, the state of affairs in a circuit containing a hysteretic medium. The analytic expression for the loop is given in parameter form by the two equations:

$$\phi = F'(t)$$

$$i = F''(t)$$

and the explicit expression would be obtained by the elimination of t between these two equations. This would be a matter of some difficulty, since obviously the loop cannot be represented by any simple expression, but it is unnecessary, since the other form is sufficient and even more convenient.

Appendix I.

Apparently the most direct method for the solution of the fundamental equation:

$$\frac{d(Li)}{dt} + \frac{R(Li)}{L} = e \;,$$

$$e = \sum (E'_n \sin npt + E''_n \cos npt)$$

would be to assume a general periodic solution:

$$i = \sum (S_n \sin npt + C_n \cos npt)$$

and by direct substitution to determine the values of S and C. We get:

$$di/dt = \sum np\{S_n \cos npt -- C_n \sin npt\}$$

since there can be no difficulty in differentiating the series term by term on account of the physical meaning of the derivative. Substituting these values and putting for brevity

$$A = k\,c\,\epsilon -- 2\,p\,k \;,$$

$$B = k\,c\,\epsilon_2 \;, \quad a = R/L \;.$$

we get the following identity:

$$\sum np\Big\{ S_n\big[\cos(n-2)pt + \cos(n+2)pt\big]$$
$$- D_n\big[\sin(n+2)pt + \sin(n-2)pt\big]$$
$$+ A\sum\Big\{ C_n\big[\cos(n+2)pt - \cos(n-2)pt\big] + C_n\big[\sin(n+2) - \sin(n-2)\big]\Big\}$$
$$+ B\sum\Big\{ L_n\big[\sin(n+2) + \sin(\text{[illegible]})\big] + L_n\,\text{[illegible]}\Big\}$$
$$+ C\sum\Big\{ L_n \sin npt + \text{[illegible]}\Big\} = \sum\Big\{ a_n \sin npt + \text{[illegible]}\Big\}$$

It is important to note that the presence of an e.m.f. harmonic of order m gives rise to current harmonics of m, m±2, and to no others. This is what has been called in the text the "mixing" of harmonics, and is the cause of the distortion.

The fact that this mixing extends only to two ~~#~~ orders above and below is the basis of the approximation made in the text. It involves a slight error in the calculation of the factor $\mathfrak{T}$ but this is of very little importance.

The last equation gives the condition that must exist if the assumed value of current is to be a solution of the original equation. This implies the following relations among the coefficients:

$$b_1 = S_1(p + k_1 p + c) + C_1(\text{[illegible]}) + S_3(3k_1 p - c) + B C_3$$

$$b_3 = 3p S_3 + c C_3 + S_1(k_1 p - c) + B C_1$$
$$+ S_5(5k_1 p + c) + B C_5$$

. . . .

$$a_1 = -p C_1 - c S_1 + C_1(k_1 p + c) - B S_1$$
$$+ C_3(k_1 p - c) - B S_3$$

$$a_3 = -3p C_3 + c S_3 + C_1(-k_1 p + c) - B S_1$$
$$+ C_5(-5k_1 p - c) - B S_5$$

. . . .

where the a's and b's are the corresponding e.m.f. amplitudes divided by the quantity Lo.

Now it will be noticed that if 2n of these equations are taken, there will be present 2n + 2 unknowns, so that these are not determinate. This

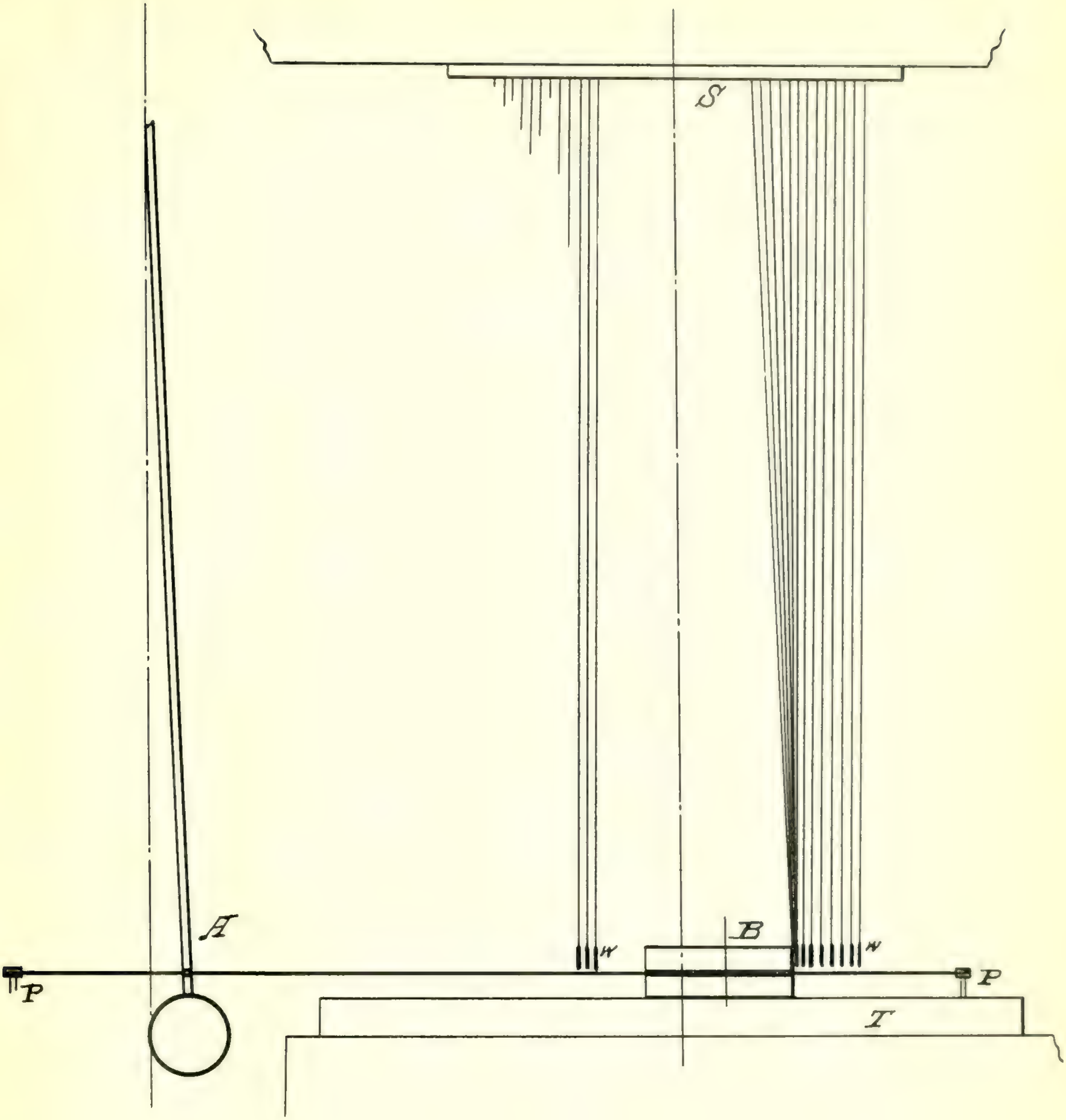
might of course have been predicted from the form of the original equation. If we may assume that one set of the higher harmonics of current vanishes, for instance S_7 and C_7, we may still not solve the set of equations, for then we shall have more than $2n$ relations among the $2n$ unknowns, which cannot in general be true. By the method of successive approximations we may arrive at the ratios of the amplitudes, and then from a knowledge of the total current and the power factor obtain approximate values of the harmonic amplitudes, but this is a tedious process not suited to practical purposes, so that the method outlined in the text is a better one.

APPENDIX II.

A MECHANICAL MODEL.

The equation satisfied by the current is of the same form as that describing the course of events in a dynamical system in which the mass is a doubly-periodic function of the time, while the impressed force is simply periodic. It is therefore possible to construct a dynamical model of the electrical system, and to the current will correspond the velocity of the mass m; to the e.m.f. Ri will correspond the resisting force, proportional to the velocity; to the impressed e.m.f., the impressed mechanical force.

A model which meets these conditions is shown in the figure. The impressed force is furnished by the pendulum A which is so massive that the system does not react upon it to modify its motion. The block B is set in motion by A and slides upon a table T of roughness varying with the position of B. The variation in mass is accomplished by adding to the mass of B that of a number of weights w, this number depending upon the position of B and hence upon the time. The suspensions of the weights must be so long that practically no restoring forces are called into play by their displacement. The phase of the pulsation


S
A
B
w
w
P
P
T

can be adjusted by moving the sliding piece S to the right or left, and that of the resistance by a similar displacement of the table T.

The connection between A and B must satisfy two conditions: it must not be rigid, since the pendulum is assumed massive; it must not be elastic, for this would introduce potential energy of strain into the system, which is excluded by the assumption of no electrostatic capacity. These conditions can be satisfied by making the connection by means of a rough and inextensible cord which passes, with friction, through the pendulum and block and is closed upon itself (back of the plane of the figure) by passing over two pulleys P.

Now the connection between the e.m.f. and the current, i.e., between the pendulum and the block, is made by means of the *magnetic field* and the conductivity of the medium, hence the analogue of the flux, ϕ, is some function of the position of a point on the cord. More exact reasoning is necessary to find the form of the function, and this can easily be given, as follows: the impressed e.m.f. is the derivative of the flux, *plus* a term which is linear in the current. Now the impressed mechanical force due to A (that is, the impressed acceleration), is the derivative of the velocity of the cord, *plus* a term depending upon the

slipping of the cord (coefficient of friction) and which is linear in the velocity of the block. Since the velocity of the block is the analogue of the current, it follows that the velocity of a point on the cord is the complete analogue of the magnetic flux, since no other quantities are needed to specify it. (More exactly, it is the momentum of the cord.)

It can be easily seen that this actually represents a hysteretic process, for the cord will lag behind the impressed force, or lead the induced force by an amount which depends upon the coefficient of friction (effective conductance representing hysteresis loss) and upon the inertia of the cord.

The block may be made to trace out the current wave by fixing a tracing point to it, in connection with a revolving drum. This will give the position of the block as function of the time. The velocity (current) as function of the time may be obtained by constructing a curve whose ordinates are the slopes of the position-time curve at each point.

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